From Reference Frame Relativity to Relativity of Mathematical Models : Relativity Formulas in a Variety of Non-Archimedean Setups

(part I)

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Abstract

Galilean Relativity and Einstein's Special and General Relativity showed that the Laws of Physics go deeper than their representations in any given reference frame. Thus covariance, or independence of Laws of Physics with respect to changes of reference frames became a fundamental principle. So far, all of that has only been expressed within one single mathematical model, namely, the traditional one built upon the usual continuum of the field $\mathbb{R}$ of real numbers, since complex numbers, finite dimensional Euclidean spaces, or infinite dimensional Hilbert spaces, etc., are built upon the real numbers. Here, following [55], we give one example of how one can go beyond that situation and study what stays the same and what changes in the Laws of Physics, when one models them within an infinitely large variety of algebras of scalars constructed rather naturally. Specifically, it is shown that the Special Relativistic addition of velocities can naturally be considered in any of infinitely many reduced power algebras, each of them containing the usual field of real numbers and which, unlike the latter,

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are non-Archimedean. The nonstandard reals are but one case of such reduced power algebras, and are as well non-Archimedean. Two surprising and strange effects of such a study of the Special Relativistic addition of velocities are that one can easily go beyond the velocity of light, and rather dually, one can as easily end up frozen in immobility, with zero velocity. Both of these situations, together with many other ones, are as naturally available, as the usual one within real numbers.

1. Introduction

There has for longer been an awareness that the exclusive use of the continuum given by the field $\mathbb{R}$ of real numbers in building up the conventional modelling of Physical space-time is in fact *not* implied by any particular Physical reason, but rather by Mathematical convenience. Details in this regard can be found in [2,3,5,10,11,21-26,31,48,49, 52,54,55] and the literature cited there.
One of the simplest and hardest arguments in this regard has been the observation that only *rational* numbers - thus in $\mathbb{Q}$ and not in the whole of $\mathbb{R}$ - can ever turn up as results of Physical measurements. And needless to say, the difference between $\mathbb{Q}$ and $\mathbb{R}$ is considerable, not least since the former is merely countable, while the latter is not, being in fact uncountable and of the power of the continuum.

Needless to say, if we want to replace $\mathbb{R}$, and thus the usual structures built upon it and used in Physics, with other scalars and corresponding structures, we still need to keep *basic algebraic* operations such as *addition, subtraction, multiplication and division*. However, such operations do *not* necessarily imply the use of fields, since they can be performed as well in the more general algebraic structures called *algebras*. And as well known, and shown in the sequel, there are *many more* algebras to use, than fields. And in fact, there are *infinitely many* such algebras which can be constructed in rather easy ways, thus making their use convenient, see [54,55].

In this way

> The issue of progressing one step ahead in *furthering the Principle of Relativity*, this time not by mere covariance

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with respect to reference frames, but by studying the possible covariance with respect to algebras of scalars in which the Laws of Physics are formulated, may appear as rather natural.

A remarkable latest contribution in this regard can be found in [32]. One of the basic deficiencies of the first von Neumann Hilbert space model of Quantum Mechanics has been the fact that observables of central importance, such a position, momentum or energy, may lack eigenstates within the respective Hilbert spaces, [51]. So far, the only rigorous mathematical approach to this deficiency has been the rigged-Hilbert space formalism. However, in this approach the respective eigenstates may still fall outside of the Hilbert spaces under consideration. It is in this regard that [32], by the use of the nonstandard real numbers, brings a convenient clarification.

However, it is important to note the following. The alternative is *not at all* restricted to either using the usual field $\mathbb{R}$ of real numbers, or instead, the nonstandard field $^*\mathbb{R}$.
Indeed, as it happens, there is naturally a far *larger* choice at disposal. Namely, one can study the use of any of infinitely many algebras, among which both the usual reals and the nonstandard ones are but particular cases. And this infinitely large class of algebras can be constructed by the *reduced power* method presented briefly in the sequel, see also [54, 55].

By the way of the latest contribution in [32], a less well known fact seems to be that, as early as 1935, von Neumann himself got disappointed in the use of Hilbert spaces in Quantum Mechanics, as mentioned in one of his letters, [51].

Recent suggestions for going beyond the exclusive use of $\mathbb{R}$ in building models of Physical space-time have been presented in [2,3,5,10,11,21,23, 26,48,49,52,54,55], as well as in the literature mentioned there.

Among others, one can envision the replacement of $\mathbb{R}$ by other fields of numbers, such as for instance, the p-adic fields $\mathbb{Q}_p$, for various prime numbers $p \in \mathbb{N}$, or even the field $^*\mathbb{R}$ of nonstandard reals.

On the other hand, as noted in [54,55], the need to use *fields* is not implied by any specific Physical reason either. Therefore, one could also employ the more general structures of *algebras*, and in this case, as recalled in [55], there is a practically unlimited, and in fact infinite variety of such algebras which can be constructed easily as *reduced powers* of $\mathbb{R}$.

It is important to note that with the conventional use of $\mathbb{R}$, one of the specific features of $\mathbb{R}$ one has to accept is that, as an ordered field, $\mathbb{R}$ is *Archimedean.* In this regard, two facts can be noted. First, the need to have the Archimedean property is again not implied by any particular Physical reason. And in fact, this property often leads to difficulties related to so called "infinities in Physics", difficulties attempted to be treated by various ad-hoc "re-normalization" procedures. On the other hand, such difficulties can easily be avoided from the start, if the use of $\mathbb{R}$ is set aside, since they simply do no longer appear in case one employs instead non-Archimedean structures, as illustrated in the better known case of nonstandard reals $^*\mathbb{R}$. Second, the alternatives suggested to $\mathbb{R}$, such as the p-adic fields $\mathbb{Q}_p$ or the infinitely large class of algebras constructed as reduced powers, turn out *not* to be Archimedean.

Consequently, lacking any known Physical reason why we should be confined to the use of Archimedean structures alone, we can equally investigate the way various Laws of Physics may take shape when non-Archimedean structures are employed.

In this regard, in the sequel, we shall study what happens to the law of *addition of velocities* in Special Relativity, when instead of the conventional Archimedean field $\mathbb{R}$ of real numbers one employs any in the infinite class of algebras of reduced powers which, as mentioned, are non-Archimedean.

One of the unexpected and strange effects of considering the Special Relativistic addition of velocities in non-Archimedean setup is that one can easily go beyond the velocity of light, and somewhat dually, one can as easily end up frozen in immobility, with zero velocity, both of these situations, together with many other ones, being as naturally

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available, as the usual one.

2. Isomorphisms of velocity addition

As shown in [53], and specified briefly in section 4 below, velocity addition in Special Relativity and Newtonian Mechanics are isomorphic as group operations.

Namely, let $c > 0$ be the velocity of light in vacuum. Then, as is well known, in the case of uniform motion along a straight line, the Special Relativistic addition $*$ of velocities is given by

(SR) $\qquad u * v = (u + v)/(1 + uv/c^2), \quad u, v \in (-c, c)$

thus the binary operation $*$ acts according to

$\qquad * : (-c, c) \times (-c, c) \longrightarrow (-c, c)$

It follows immediately that

(IS1) $\quad *$ is associative and commutative

(IS2) $\quad u * v * w = (u + v + w + uvw/c^2)/(1 + (uv + uw + vw)/c^2)$
$\qquad$ for $u, v, w \in (-c, c)$

(IS3) $\quad u * 0 = 0 * u = u, \quad u \in (-c, c)$

(IS4) $\quad u * (-u) = (-u) * u = 0, \quad u \in (-c, c)$

(IS5) $\quad \partial/\partial u(u * v) = (1 - v^2/c^2)/(1 + uv/c^2)^2 > 0, \quad u, v \in (-c, c)$

(IS6) $\quad \lim_{u, v \to c} u * v = c, \quad \lim_{u, v \to -c} u * v = -c$

Therefore

(IS7) $\quad ((-c, c), *)$ is a commutative group with the neutral
$\qquad$ element 0, while $-u$ is the inverse element of $u \in (-c, c)$

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3. Velocity addition in Newtonian Mechanics

As is well known, in the case of uniform motion along a straight line, the addition of velocities in Newtonian Mechanics is given by

(NM) $x + y, \quad x, y \in \mathbb{R}$

thus it is described by the usual additive group $(\mathbb{R}, +)$ of the real numbers, a group which is of course commutative, with the neutral element 0, while $-x$ is the inverse element of $x \in \mathbb{R}$.

4. Isomorphisms of the two groups of velocity addition

As shown in [53], the following hold.

(IS8) $((-c, c), *)$ and $(\mathbb{R}, +)$ are isomorphic groups through the mappings

(IS8.1) $\alpha : (-c, c) \longrightarrow \mathbb{R}$, where

$$\alpha(u) = k \ln((c + u)/(c - u)), \quad u \in (-c, c)$$

and

(IS8.2) $\beta : \mathbb{R} \longrightarrow (-c, c)$, where

$$\beta(x) = c(e^{x/k} - 1)/(e^{x/k} + 1), \quad x \in \mathbb{R}$$

with

(IS8.3) $k = c^2 \alpha'(0) > 0$

(IS9)   both α and β are strictly increasing mappings

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5. Note

The Special Relativistic addition $*$ of velocities in (SR) is in fact well defined not only for pairs of velocities

$$(u, v) \in (-c, c) \times (-c, c)$$

but also for the *larger* set of pairs of velocities

$$(u, v) \in [-c, c] \times [-c, c], \quad uv \neq -c^2$$

This corresponds to the fact that in Special Relativity the velocity c of light in vacuum is supposed to be attainable, namely, by light itself in vacuum.

On the other hand, the Newtonian addition $+$ of velocities (NM) does of course only make sense Physically for

$$(x, y) \in \mathbb{R} \times \mathbb{R}$$

since infinite velocities are not supposed to be attainable Physically.

As for the group isomorphisms α and β, they only generate mappings between pairs of velocities in

$$(-c, c) \times (-c, c) \xrightarrow{\alpha \times \alpha} \mathbb{R} \times \mathbb{R}$$

and

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\beta \times \beta} (-c, c) \times (-c, c)$$

thus they do *not* cover the cases of addition $u * v$ of special relativistic velocities $u = -c$ and $v < c$, or $-c < u$, and $v = c$.

Consequently, in spite of the group isomorphisms α and β, there is an *essential* difference between the addition of velocities in Special Relativity, and on the other hand, Newtonian Mechanics. Indeed, in the latter case, the addition $+$ is defined on the *open* set $\mathbb{R} \times \mathbb{R}$, while in

the former case the addition $*$ is defined on the set

$$\{ (u, v) \mid -c \le u, v \le c, \quad uv \ne -c^2 \}$$

which is *neither open, nor closed.*

6. Reduced Power Algebras

We shall now show how the group isomorphisms (IS8.2), (IS8.3) can naturally be extended to reduced power algebras.

First, for convenience, let us recall briefly the general method for constructing an infinitely large class of algebras obtained as reduced powers, [55]. This *reduced power construction*, in its more general forms, is one of the fundamental tools in Model Theory, [20,58]. Historically, even if only in a particular case and in an informal manner, it can be traced back to its use in the 19th century in the classical Cauchy-Bolzano construction of the field $\mathbb{R}$ of real numbers from the set $\mathbb{Q}$ of rational ones. Various other familiar instances of the reduced power construction in modern Mathematics can often be encountered, for instance, when completing metric spaces, or in general, uniform topological spaces.

Let Λ be any infinite set, then the power $\mathbb{R}^\Lambda$ is in a natural way an associative and commutative *algebra*. Namely, the elements $\xi \in \mathbb{R}^\Lambda$ can be seen as mappings $\xi : \Lambda \longrightarrow \mathbb{R}$, and as such, they can be added to, and multiplied with one another point-wise. Namely, if $\xi, \xi' : \Lambda \longrightarrow \mathbb{R}$, then

$$(\xi + \xi')(\lambda) = \xi(\lambda) + \xi'(\lambda), \quad \lambda \in \Lambda$$

$$(\xi . \xi)(\lambda) = \xi(\lambda) . \xi'(\lambda), \quad \lambda \in \Lambda$$

In the same way, the elements $\xi \in \mathbb{R}^\Lambda$ can be multiplied with scalars from $\mathbb{R}$, namely

$$(a . \xi)(\lambda) = a . \xi(\lambda), \quad a \in \mathbb{R}, \quad \lambda \in \Lambda$$

The well known remarkable fact connected with such a power algebra $\mathbb{R}^\Lambda$ is that there is a one-to-one correspondence between the *proper ideals* in it, and on the other hand, the *filters* on the infinite set Λ, see for instance [54,55] and the literature cited there. Indeed, this one-to-one correspondence operates as follows

(6.1)
$$\mathcal{I} \longmapsto \mathcal{F}_\mathcal{I} = \{ Z(\xi) \mid \xi \in \mathcal{I} \}$$
$$\mathcal{F} \longmapsto \mathcal{I}_\mathcal{F} = \{ \xi \in \Lambda \longrightarrow \mathbb{R} \mid Z(\xi) \in \mathcal{F} \}$$

where $\mathcal{I}$ is an ideal in $\mathbb{R}^\Lambda$, $\mathcal{F}$ is a filter on Λ, while for $\xi \in \Lambda \longrightarrow \mathbb{R}$, we denoted $Z(\xi) = \{\lambda \in \Lambda \mid \xi(\lambda) = 0\}$, that is, the zero set of ξ. As is known, the critical part in (6.1) is that $\mathcal{F}_\mathcal{I}$ constitutes a filter on Λ, see the proof of (3.7) in [54] for details.

The great practical *advantage* of the one-to-one correspondence between the proper ideals in $\mathbb{R}^\Lambda$, and on the other hand, the filters on the infinite set Λ is that the latter are much *simpler* mathematical structures. Furthermore, the specific way reduced powers are constructed, see (6.4), (6.5) below, brings in the power of clarity and simplicity which made Model Theory such an important branch of modern Mathematics.

However, it is important to note that, fortunately, no knowledge of Model Theory is needed in order to be able to make full use of the reduced power algebras. Indeed, a usual first course in Algebra, covering such issues as groups, quotient groups, rings and ideals is sufficient. In this regard, Model Theory comes in only in order to motivate and highlight the naturalness of the construction which leads to reduced power algebras.

Here one further fact should be noted. As is well known, the nonstandard reals $^*\mathbb{R}$ can also be constructed as reduced powers. However, their respective construction is in a way an extreme case, since it uses the special class of free ultrafilters. An essential resulting aspect is the so called *transfer property*, with the accompanying discrimination between internal and external entities, which introduces a whole host of technical complications to deal with, in order to be able to benefit

fully from the power of Nonstandard Analysis.

In this regard, the use of free ultrafilters is needed in the construction of nonstandard reals $^*\mathbb{R}$ only in order to obtain them as a *totally ordered field.* Otherwise, if we are ready to use algebras which are not fields, nor totally ordered, we can employ the much larger class of filters, instead of ultrafilters. Anyhow, even in the case of the nonstandard reals we end up with the non-Archimedean property.

Consequently, the general form of the reduced power construction used here, as well as in [54, 55], does not restrict itself to ultrafilters, and thus avoids the mentioned technical difficulties related to nonstandard transfer, yet it can benefit from much of the power, clarity and simplicity familiar in Model Theory.

Important properties of the one-to-one correspondence in (6.1) are as follows. Given two ideals $\mathcal{I}, \mathcal{J}$ in $\mathbb{R}^\Lambda$, and two filters $\mathcal{F}, \mathcal{G}$ on Λ, we have

$$\text{(6.2)} \qquad \begin{aligned} \mathcal{I} \subseteq \mathcal{J} &\implies \mathcal{F}_\mathcal{I} \subseteq \mathcal{F}_\mathcal{J} \\ \mathcal{F} \subseteq \mathcal{G} &\implies \mathcal{I}_\mathcal{F} \subseteq \mathcal{I}_\mathcal{G} \end{aligned}$$

Furthermore, the correspondences in (6.1) are idempotent when iterated, namely

$$\text{(6.3)} \qquad \begin{aligned} \mathcal{I} &\longmapsto \mathcal{F}_\mathcal{I} \longmapsto \mathcal{I}_{\mathcal{F}_\mathcal{I}} = \mathcal{I} \\ \mathcal{F} &\longmapsto \mathcal{I}_\mathcal{F} \longmapsto \mathcal{F}_{\mathcal{I}_\mathcal{F}} = \mathcal{F} \end{aligned}$$

It follows that every *reduced power algebra*

$$\text{(6.4)} \quad A = \mathbb{R}^\Lambda / \mathcal{I}$$

where $\mathcal{I}$ is a proper ideal in $\mathbb{R}^\Lambda$, is of the form

$$\text{(6.5)} \quad A = A_\mathcal{F} \overset{def}{=} \mathbb{R}^\Lambda / \mathcal{I}_\mathcal{F}$$

for a suitable unique filter $\mathcal{F}$ on Λ.

We shall call Λ the *index set* of the reduced power algebra $A_\mathcal{F} = \mathbb{R}^\Lambda / \mathcal{I}_\mathcal{F}$, while $\mathcal{F}$ will be called the *generating filter* which, we recall, is a filter

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on that index set.

Obviously, we can try to relate various reduced power algebra $A_{\mathcal{F}} = \mathbb{R}^\Lambda / \mathcal{I}_{\mathcal{F}}$ to one another, according to the two corresponding parameters which define them, namely, their infinite index sets Λ and their generating filters $\mathcal{F}$.

We start here by relating them with respect to the latter. Namely, a direct consequence of the second implication in (6.2) is the following one. Given two filters $\mathcal{F} \subseteq \mathcal{G}$ on Λ, we have the *surjective algebra homomorphism*

$$(6.6) \quad A_{\mathcal{F}} \ni \xi + \mathcal{I}_{\mathcal{F}} \ \longmapsto \ \xi + \mathcal{I}_{\mathcal{G}} \in A_{\mathcal{G}}$$

This obviously means that the algebra $A_{\mathcal{G}}$ is *smaller* than the algebra $A_{\mathcal{F}}$, the precise meaning of it being that

$$(6.6^*) \quad A_{\mathcal{G}} \ \text{and} \ A_{\mathcal{F}}/(\mathcal{I}_{\mathcal{G}}/\mathcal{I}_{\mathcal{F}}) \ \text{are isomorphic algebras}$$

which follows from the so called *third isomorphism theorem for rings*, a classical result of Algebra.

Here we note that in the particular case when the filter $\mathcal{F}$ on Λ is generated by a nonvoid subset $I \subseteq \Lambda$, that is, when we have

$$(6.7) \quad \mathcal{F} = \{\, J \subseteq \Lambda \mid J \supseteq I \,\}$$

then it follows easily that

$$(6.8) \quad A_{\mathcal{F}} \ = \ \mathbb{R}^I$$

which means that we do *not* in fact have a reduced power algebra, but only a power algebra.

For instance, in case I is finite and has $n \geq 1$ elements, then $A_{\mathcal{F}} = \mathbb{R}^n$ is in fact the usual n-dimensional Euclidean space.

Consequently, in order to avoid such a degenerate case of reduced

power algebras, we have to avoid the filters of the form (6.7). This can be done easily, since such filters are obviously characterized by the property

$$(6.9) \quad \bigcap_{J \in \mathcal{F}} J \;=\; I \neq \phi$$

It follows that we shall only be interested in filters $\mathcal{F}$ on Λ which satisfy the condition

$$(6.10) \quad \bigcap_{J \in \mathcal{F}} J \;=\; \phi$$

or equivalently

$$\forall \; \lambda \in \Lambda \; :$$

$$(6.11) \quad \exists \; J_\lambda \in \mathcal{F} \; :$$

$$\lambda \notin J_\lambda$$

which is further equivalent with

$$\forall \; I \subset \Lambda, \;\; I \;\; \text{finite} \; :$$
$$(6.12)$$
$$\Lambda \setminus I \in \mathcal{F}$$

We recall now that the *Frechét filter* on Λ is given by

$$(6.13) \quad \mathcal{F}re(\Lambda) \;=\; \{ \, \Lambda \setminus I \;\mid\; I \subset \Lambda, \;\; I \;\; \text{finite} \, \}$$

In this way, condition (6.10) - which we shall ask from now on about all filters $\mathcal{F}$ on Λ - can be written equivalently as

$$(6.14) \quad \mathcal{F}re(\Lambda) \;\subseteq\; \mathcal{F}$$

This in particular means that

$$\forall \; I \in \mathcal{F} \; :$$
$$(6.14^*)$$
$$I \;\; \text{is infinite}$$

Indeed, if we have a finite $I \in \mathcal{F}$, then $\Lambda \setminus I \in \mathcal{F}re(\Lambda)$, hence (6.14) gives $\Lambda \setminus I \in \mathcal{F}$. But $I \cap (\Lambda \setminus I) = \phi$, and one of the axioms of filters is contradicted.

In view of (6.6), it follows that all reduced power algebras considered from now on will be *homomorphic images* of the reduced power algebra $A_{\mathcal{F}re(\Lambda)}$, through the surjective algebra homomorphisms

$$(6.15) \quad A_{\mathcal{F}re(\Lambda)} \ni \xi + \mathcal{I}_{\mathcal{F}re(\Lambda)} \quad \longmapsto \quad \xi + \mathcal{I}_\mathcal{F} \in A_\mathcal{F}$$

or in view of (6.6*), we have the *isomorphic algebras*

$$(6.15^*) \quad A_\mathcal{F}, \quad A_{\mathcal{F}re(\Lambda)}/(\mathcal{I}_\mathcal{F}/\mathcal{I}_{\mathcal{F}re(\Lambda)})$$

Let us note that the *nonstandard reals* $^*\mathbb{R}$ are a particular case of the above reduced power algebras (6.4). Indeed, $^*\mathbb{R}$ can be defined by using *free ultrafilters* $\mathcal{F}$ on Λ, that is, ultrafilters which satisfy (6.10), or equivalently (6.14).

We note that the field of real numbers $\mathbb{R}$ can be embedded naturally in each of the reduced power algebras (6.4), by the *injective algebra homomorphism*

$$(6.16) \quad \mathbb{R} \ni x \quad \longmapsto \quad \xi_x + \mathcal{I} \in A$$

where $\xi_x(\lambda) = x$, for $\lambda \in \Lambda$. Indeed, if $\xi_x \in \mathcal{I}$ and $\xi \neq 0$, then the ideal $\mathcal{I}$ must contain x_1, which means that it is *not* a proper ideal, thus contradicting the assumption on it.

For simplicity of notation, we may write $\xi_x = x$, for $x \in \mathbb{R}$, thus (6.16) can take the form

$$(6.17) \quad \mathbb{R} \ni x \quad \longmapsto \quad x = \xi_x + \mathcal{I} \in A$$

which in view of the injectivity of this mapping, we may further simplify to

$$(6.18) \quad \mathbb{R} \ni x \quad \longmapsto \quad x \in A$$

in other words, to the *algebra embedding*

$$(6.18^*) \quad \mathbb{R} \subsetneqq A$$

There is also the issue to relate reduced power algebras corresponding to different *index sets*. Namely, let $\Lambda \subseteq \Gamma$ be two infinite index sets. Then we have the obvious *surjective algebra homomorphism*

$$(6.19) \quad \mathbb{R}^\Gamma \ni \xi \longmapsto \xi|_\Lambda \in \mathbb{R}^\Lambda$$

since the elements $\xi \in \mathbb{R}^\Gamma$ can be seen as mappings $\xi : \Gamma \longrightarrow \mathbb{R}$. Consequently, given any ideal $\mathcal{I}$ in $\mathbb{R}^\Gamma$, we can associate with it the ideal in $\mathbb{R}^\Lambda$, given by

$$(6.20) \quad \mathcal{I}|_\Lambda = \{ \xi|_\Lambda \mid \xi \in \mathcal{I} \}$$

As it happens, however, such an ideal $\mathcal{I}|_\Lambda$ need not always be a proper ideal in $\mathbb{R}^\Lambda$, even if $\mathcal{I}$ is a proper ideal in $\mathbb{R}^\Gamma$. For instance, if we take $\gamma \in \Gamma \setminus \Lambda$, and consider the proper ideal in $\mathbb{R}^\Gamma$ given by $\mathcal{I} = \{\xi \in \mathbb{R}^\Gamma \mid \xi(\gamma) = 0 \}$, then we obtain $\mathcal{I}|_\Lambda = \mathbb{R}^\Lambda$, which is not a proper ideal in $\mathbb{R}^\Lambda$.

We can avoid that difficulty by noting the following. Given a filter $\mathcal{F}$ on Γ which satisfies (6.14), that is, $\mathcal{F}re(\Gamma) \subseteq \mathcal{F}$, then

$$(6.21) \quad \mathcal{F}|_\Lambda = \{ I \cap \Lambda \mid I \in \mathcal{F} \}$$

satisfies the corresponding version of (6.14), namely $\mathcal{F}re(\Lambda) \subseteq \mathcal{F}|_\Lambda$. Indeed, let us take $J \subseteq \Lambda$ such that $\Lambda \setminus J$ is finite. Then clearly $\Gamma \setminus (J \cup (\Gamma \setminus \Lambda))$ is finite, hence $J \cup (\Gamma \setminus \Lambda) \in \mathcal{F}$. However, $J = (J \cup (\Gamma \setminus \Lambda)) \cap \Lambda)$, thus $J \in \mathcal{F}|_\Lambda$.
Now in order for $\mathcal{F}|_\Lambda$ to be a filter on Λ, it suffices to show that $\phi \notin \mathcal{F}|_\Lambda$. Assume on the contrary that for some $I \in \mathcal{F}$ we have $I \cap \Lambda = \phi$, then $I \subseteq \Gamma \setminus \Lambda$, thus $\Lambda \notin \mathcal{F}$.

It follows that

(6.22) $\mathcal{F}|_\Lambda$ is a filter on Λ which satisfies (6.14) $\Longleftrightarrow$ $\Lambda \in \mathcal{F}$

In view of (6.19) - (6.22), for every filter $\mathcal{F}$ on Γ, such that

(6.23) $\Lambda \in \mathcal{F}$

we obtain the *surjective algebra homomorphism*

(6.24) $A_\mathcal{F} \ = \ \mathbb{R}^\Gamma / \mathcal{I}_\mathcal{F} \ni \xi + \mathcal{I}_\mathcal{F} \ \longmapsto \ \xi|_\Lambda + \mathcal{I}_{\mathcal{F}|_\Lambda} \in A_{\mathcal{F}|_\Lambda} \ = \ \mathbb{R}^\Lambda / \mathcal{I}_{\mathcal{F}|_\Lambda}$

and in particular, we have the following relation between the respective proper ideals

(6.25) $(\mathcal{I}_\mathcal{F})|_\Lambda \ = \ \mathcal{I}_{\mathcal{F}|_\Lambda}$

7. Zero Divisors and the Archimedean Property

It is an elementary fact of Algebra that a quotient algebra (6.4) has *zero divisors*, unless the ideal $\mathcal{I}$ is *prime*. A particular case of that is when a quotient algebra (6.4) is a *field*, which is characterized by the ideal $\mathcal{I}$ being *maximal*. And in view of (6.5), (6.2), this means that the filter $\mathcal{F}$ generating such an ideal must be an *ultrafilter*, see for details [54,55].

On the other hand, *none* of the reduced power algebras (6.5) which correspond to filters satisfying (6.14) are *Archimedean*. And that includes the nonstandard reals $^*\mathbb{R}$ as well.

In this regard, let us first we note that on reduced power algebras (6.5), one can naturally define a *partial order* as follows. Given two elements $\xi + \mathcal{I}_\mathcal{F}, \ \eta + \mathcal{I}_\mathcal{F} \in A_\mathcal{F} = \mathbb{R}^\Lambda / \mathcal{I}_\mathcal{F}$, we define

(7.1) $\xi + \mathcal{I}_\mathcal{F} \ \leq \ \eta + \mathcal{I}_\mathcal{F} \ \Longleftrightarrow \ \{ \lambda \in \Lambda \ | \ \xi(\lambda) \ \leq \ \eta(\lambda) \} \in \mathcal{F}$

Now, with this partial order, the algebra $A_\mathcal{F}$ would be *Archimedean*, if and only if

$$\exists \ v + \mathcal{I}_\mathcal{F} \in A_\mathcal{F}, \ v + \mathcal{I}_\mathcal{F} \geq 0 \ :$$

$$\forall \ \xi + \mathcal{I}_\mathcal{F} \in A_\mathcal{F}, \ \xi + \mathcal{I}_\mathcal{F} \geq 0 \ :$$

(7.2)

$$\exists \ n \in \mathbb{N} \ :$$

$$\xi + \mathcal{I}_\mathcal{F} \ \leq \ nv + \mathcal{I}_\mathcal{F}$$

However, in view of (6.14), we can take an infinite $I \in \mathcal{F}$. Thus we can define a mapping $\omega : \Lambda \longrightarrow \mathbb{R}$ which is unbounded from above on I. And in this case taking $\xi + \mathcal{I}_\mathcal{F} = (v + \omega) + \mathcal{I}_\mathcal{F}$, it follows easily that condition (7.2) is not satisfied.

We note that the reduced power algebras $A_\mathcal{F}$ in (6.5) are Archimedean only in the degenerate case (6.7), (6.8), when in addition the respective sets I are *finite*, thus as noted, the respective algebras reduce to finite dimensional Euclidean spaces.

Further we note that the partial order (7.1) on the algebras $A_\mathcal{F}$ is in general *not* a total order. In this regard, we have

Proposition 1.

Let $\mathcal{F}$ be a filter on Λ satisfying (6.14). Then the partial order (7.1) is a total order on the reduced power algebra $A_\mathcal{F}$, if and only if $\mathcal{F}$ is an ultrafilter. In that case we have $A_\mathcal{F} = {}^*\mathbb{R}$, that is, the reduced power algebra $A_\mathcal{F}$ is the field ${}^*\mathbb{R}$ of nonstandard reals.

Proof.

Let us take any partition $\Lambda = \Lambda_0 \cup \Lambda_1$ into two infinite subsets and take $\xi_0, \ \xi_1 : \Lambda \longrightarrow \mathbb{R}$ as the characteristic functions of Λ_0 and Λ_1, respectively. Then obviously

(7.3) $\quad \{\lambda \in \Lambda \mid \xi_0(\lambda) < \xi_1(\lambda)\} = \Lambda_1, \quad \{\lambda \in \Lambda \mid \xi_0(\lambda) > \xi_1(\lambda)\} = \Lambda_0$

and in general $\Lambda_0, \ \Lambda_1 \notin \mathcal{F}$, like for instance, when $\mathcal{F} = \mathcal{F}re(\Lambda)$. Thus in view of (7.1), in general, we cannot have in $A_\mathcal{F}$ either of the in-

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equalities

$$(7.4) \quad \xi_0 + \mathcal{I}_\mathcal{F} \ \leq \ \xi_1 + \mathcal{I}_\mathcal{F}, \quad \xi_0 + \mathcal{I}_\mathcal{F} \ \geq \ \xi_1 + \mathcal{I}_\mathcal{F}$$

Clearly, no filter on Λ can simultaneously contain both Λ_0 and Λ_1, thus both of the above inequalities (7.4) can never hold simultaneously.

However, in case $\mathcal{F}$ is an ultrafilter satisfying (6.14), thus we are in the particular situation when $A_\mathcal{F} = {}^*\mathbb{R}$, that is, the reduced power algebra $A_\mathcal{F}$ is the field ${}^*\mathbb{R}$ of nonstandard reals, then according to a property of ultrafilters, we must have either $\Lambda_0 \in \mathcal{F}$, or $\Lambda_1 \in \mathcal{F}$. Therefore, one and only one of the above two inequalities in (7.4) holds.

Conversely, let (7.1) be a total order on $A_\mathcal{F}$. Then one and only one of the inequalities (7.4) must hold. Let us assume that it is the case of the first one of them. Then (7.1), (7.3) imply that $\Lambda_1 \in \mathcal{F}$. Obviously, in the other case we obtain that $\Lambda_0 \in \mathcal{F}$.

In this way, whenever we are given a partition $\Lambda = \Lambda_0 \cup \Lambda_1$ into two infinite subsets, one of them must belong to the filter $\mathcal{F}$.
If on the other hand, in the partition $\Lambda = \Lambda_0 \cup \Lambda_1$ one of the sets is finite, then in view of (6.14) the other must belong to the filter $\mathcal{F}$. And since Λ is supposed to be infinite, both sets in the partition cannot be finite.

Thus we can conclude that the filter $\mathcal{F}$ is indeed an ultrafilter on Λ, since its above property related to partition characterizes ultrafilters.

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